Comment on "On Existence of a Biorthonormal Basis Composed of Eigenvectors of Non-Hermitian Operators [quant-ph/0603075]"

Ali Mostafazadeh

Department of Mathematics, Koç University,
34450 Sariyer, Istanbul, Turkey
amostafazadeh@ku.edu.tr

Abstract

We point out that T. Tanaka's recent criticism [quant-ph/0603075] of the results of J. Math. Phys. **43**, 3944 (2002) [math-ph/0203005] is based on an assumption which was never made in the latter paper, namely that the diagonalizability of an operator implies that it is normal. Therefore, Tanaka's objections regarding this paper are not valid.

The definition of a diagonalizable operator acting in an infinite-dimensional vector space is not usually given in textbooks on linear algebra or operator theory. This was the reason I hesitated to use this term in [1, 2] and indeed it was upon the insistence of the referee of [3], who had found the repeated statement of the assumption of the "existence of a complete biorthonormal basis of eigenvectors" too lengthly, that I decided to use the term "diagonalizable" in [3]. According to [3, 4], a linear operator H acting in a separable Hilbert space and having a discrete spectrum is called *diagonalizable* if there are eigenvectors ψ_n of H and ϕ_n of $H^\dagger$ that form a complete biorthonormal basis (or system) $\{\psi_n, \phi_n\}$, i.e., they satisfy

$$\langle\psi_n|\phi_m\rangle = \delta_{mn}, \qquad \sum_n |\psi_n\rangle\langle\phi_n| = \sum_n |\phi_n\rangle\langle\psi_n| = 1. \tag{1}$$

Nowhere in this definition is it assumed that the operator is normal. A normal operator, in finite-dimensions with no extra conditions and in infinite-dimensions with appropriate extra conditions, admit a diagonal matrix representation in some orthonormal basis. This is usually called "diagonalizability by a unitary transformation." The operators considered in [3] are not assumed to be normal operators and the related remarks of Tanaka [5] do not hold. For example, according to the above definition of a diagonalizable operator that is used in [3], the

operator $A:\mathbb{C}^2\to\mathbb{C}^2$ that is represented in the standard basis of $\mathbb{C}^2$ by the matrix

$$\left(\begin{array}{cc} 1 & 2 \\ 0 & 3 \end{array}\right)$$

is certainly diagonalizable, for the following vectors (also represented in the standard representation of $\mathbb{C}^2$) form a nontrivial biorthonormal basis $\{\psi_n,\phi_n\}$ satisfying all the above-mentioned properties.

$$\psi_1=\left(\begin{array}{c} 1 \\ 0 \end{array}\right),\qquad \psi_2=\left(\begin{array}{c} 1 \\ 1 \end{array}\right),\qquad \phi_1=\left(\begin{array}{c} 1 \\ -1 \end{array}\right),\qquad \phi_2=\left(\begin{array}{c} 0 \\ 1 \end{array}\right).$$

This example clearly contradicts Tanaka's following claim [5]: "*the operators which satisfy the assertion stated in Ref. [14]" (which is Ref. [3] of the present paper) "with a non-trivial biorthonormal basis must fall into, if exist, a quite peculiar class of linear operators. It should be noted in particular that there is no such operators in finite-dimensional spaces.*" This claim seems to be the result of the misunderstanding that diagonalizable operators are necessarily normal, an assumption which has never been made in [3, 4] and other papers of mine and as far as I know of others on the subject.

In conclusion, I should like to emphasize that in a quantum theory fulfilling the measurement axiom the diagonalizability (as defined above and in [3]) of the operators that are to be identified with the observables is an absolutely necessary condition [6]. The situation is just the opposite for normal operators that fail to be self-adjoint. Such operators have non-real spectra and cannot be employed as observables in a unitary quantum system. A detailed discussion of diagonalizable operators and biorthonormal systems emphasizing the mathematically rigorous results as well as the physical relevance will be given in [7].

References

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